A Wavelet-Galerkin Scheme for the Navier-Stokes equations

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June 7, 2006

Abstract

We propose a Wavelet-Galerkin scheme for the stationary Navier-Stokes equations based on the application of interpolating wavelets.

To overcome the problems of nonlinearity, we apply the machinery of interpolating wavelets presented in [10] and [13] in order to obtain problem-adapted quadrature rules. Finally, we apply Newton's method to approximate the solution in the given ansatz space, using as *inner* solver a steepest descendent scheme. To obtain approximations of a higher accuracy, we apply our scheme in a multi-scale context. Special emphasize will be given for the convergence of the scheme and wavelet preconditioning.

MSC 2000: 46E35, 65N30, 41A17, 76D05, 65F35
keywords: Galerkin scheme, wavelets, multiscale and multilevel methods, saddle-point problems, Navier-Stokes equation, Newton scheme, preconditioning

1 Introduction

During the last decade, wavelet analysis has become a field of increasing importance in the numerical treatment of partial differential equations and integral

*supported by *Fundação para Ciência e Tecnologia* under PhD-grant no. SFRH/BD/17657/2004

†Research (partially) supported by *Unidade de Investigação Matemática e Aplicações* of the University of Aveiro.

equations, see, e.g., [2, 11, 12, 14, 16, 17, 18, 20]. The advantages of wavelet methods can be described as follows. It turns out that a simple diagonal scaling applied to the stiffness matrices relative to the wavelet bases suffices to produce uniformly bounded condition numbers. Moreover, for a wide class integral and pseudo-differential operators the stiffness matrix relative to wavelet bases can be shown to be sufficiently close to sparse matrices so that sparse solvers such as conjugate gradient schemes achieve optimal complexity under minimal regularity assumptions.

So far all these potential advantages of wavelet methods have been exploited in many settings and yield powerful convergent Galerkin schemes. The most impressive results were obtained for self-adjoint and saddle point problems. For these problems it has been possible to derive optimal adaptive wavelet schemes [2, 11, 12, 14, 21].

It is therefore natural to explore the potential of such techniques for nonlinear problems. A first strategy to attack was proposed in [12]. After transforming the equation to a well-posed ℓ_2-problem, a locally convergent iterative scheme is applied to the (infinite dimensional) problem. The involved operators are adaptively evaluated within suitable updated error tolerances. However, in this paper, we proceed in some sense, other way around. By using the classical Galerkin approach, we project our problem onto an increasing sequence of approximation spaces spanned by wavelets. Then the computation of the actual Galerkin approximation requires the solution of nonlinear equations in a (finite dimensional) space. Althought the first approach seems to be more powerful, at least in the long run, we are now interested in develop stable numerical schemes and wavelet preconditioning for nonlinear equations. As a typical example we will focus here on the Navier-Stokes equation as a model for the motion of an incompressible, viscous fluid in a d-dimensional domain $\Omega \subset \mathbb{R}^d$, where $d = 2$ or $d = 3$ are of the primary interest. Stable numerical schemes for the Navier-Stokes equations for large viscosities (i.e. small Reynolds numbers) can be derived (see e.g. [4]). In this case, preconditioning results as derived in e.g. [19, 29] carry over without serious difficulties. For small viscosities, the derivation of stable numerical schemes and, moreover, nice preconditioners is slightly complicated due the lack of stability and global convergent results.

After projecting our problem onto the (finite dimensional) wavelet spaces, we are faced with two basic problems, namely how to solve the resulting nonlinear equation and how to evaluate the nonlinear functionals of wavelet expansions induced by the addition of a nonlinear perturbation. For this purpose, we will implement the approach proposed in [10].

This approach is based on a treatment of the nonlinear equation by a version of Newton's method and the evaluation of nonlinear terms is attacked by a wavelet variant of the classical "knot oriented quadrature rules" by using interpolating scaling functions. In the end preconditioners are derived for the presented version of Newton's method.

2 The Scope of Problems

We consider the following adaptation of the Navier-Stokes equations

$$\begin{aligned} \tau\mathbf{u} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot \operatorname{grad})\mathbf{u} + \operatorname{grad} p &= \mathbf{f} && \text{in } \Omega, && (1)\\ \operatorname{div}\mathbf{u} &= 0 && \text{in } \Omega, && (2)\\ \mathbf{u} &= \mathbf{0} && \text{on } \Gamma = \partial\Omega && (3) \end{aligned}$$

with the additional condition

$$\int_\Omega p dx = 0. \quad (4)$$

Hereby, $\mathbf{u}$ denotes the velocity of the fluid, p the hydrostatic pressure, ν is the kinematic viscosity (equivalent to the inverse of the Reynolds number) and $\mathbf{f}$ the vector of the external forces. The above adaptation usually results from an implicit time discretization with time step $\tau > 0$, as in the application of the Rothe's method to the nonstationary Navier-Stokes equations.

Throughout this work, we use boldface type to denote vector-valued functions having d components. The corresponding function spaces will be presented also in boldface type. For simplicity, we shall use the same inner product and norm notation for vector field function spaces. In addition to the usual function spaces we will introduce the function space

$$L_{2,0}(\Omega) := \left\{ q \in L_2(\Omega) : \int_\Omega q dx = 0 \right\}$$

which is isomorphic to the quotient space $L_2(\Omega)/\mathbb{R}$ (see e.g. [5, 32]).

Introducing the notation

$$\operatorname{grad}\mathbf{u}\cdot\operatorname{grad}\mathbf{v} := \sum_{i=1}^{d} \operatorname{grad} u^i \cdot \operatorname{grad} v^i = \sum_{i,j=1}^{d} \frac{\partial u^i}{\partial x_j}\frac{\partial v^i}{\partial x_j}$$

and the multi-linear forms

$$\begin{aligned} a_{\tau,\nu}(\mathbf{u},\mathbf{v}) &:= \int_\Omega (\tau\mathbf{u}\cdot\mathbf{v} + \nu \operatorname{grad}\mathbf{u}\cdot\operatorname{grad}\mathbf{v}) dx\\ b(\mathbf{v},q) &:= -\int_\Omega (\operatorname{div}\mathbf{v}) q dx = \int_\Omega \mathbf{v}\cdot\operatorname{grad} q dx\\ c(\mathbf{u},\mathbf{v},\mathbf{w}) &:= \int_\Omega (\mathbf{u}\cdot\operatorname{grad})\mathbf{v}\cdot\mathbf{w} dx = \sum_{m,n=1}^{d}\int_\Omega u^m \frac{\partial v^n}{\partial x_m} w^n dx \end{aligned}$$

we obtain the *mixed* formulation: find $(\mathbf{u},p) \in \mathbf{H}_0^1(\Omega)\times L_{2,0}(\Omega)$ such that

$$\begin{aligned} a_{\tau,\nu}(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) + c(\mathbf{u},\mathbf{u},\mathbf{v}) &= \int_\Omega \mathbf{f}\cdot\mathbf{v} dx && \forall_{\mathbf{v}\in\mathbf{H}_0^1(\Omega)} && (5)\\ b(\mathbf{u},q) &= 0 && \forall_{q\in L_{2,0}(\Omega)}. && (6) \end{aligned}$$

This formulation can be rewritten in the operator form: find $(\mathbf{u}, p) \in \mathbf{H}_0^1(\Omega) \times L_2(\Omega) =: \mathcal{H}$ such that

$$\mathcal{F}_{\tau,\nu}(\mathbf{u}, p) := \begin{pmatrix} A_{\tau,\nu} & B' \\ B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} + \begin{pmatrix} C(\mathbf{u}) - \mathbf{f} \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 0 \end{pmatrix} \tag{7}$$

where

$$\begin{array}{rrl} A_{\tau,\nu} : \mathbf{H}_0^1(\Omega) \to \mathbf{H}^{-1}(\Omega) & \langle A_{\tau,\nu}\mathbf{w}, \mathbf{v}\rangle_{\mathbf{H}^{-1}(\Omega)\times\mathbf{H}_0^1(\Omega)} = a_{\tau,\nu}(\mathbf{w}, \mathbf{v}), & \forall_{\mathbf{v}\in\mathbf{H}_0^1(\Omega)} \\ B : \mathbf{H}_0^1(\Omega) \to L_2(\Omega) & \langle B\mathbf{w}, q\rangle_{\mathbf{H}^{-1}(\Omega)\times\mathbf{H}_0^1(\Omega)} = b(\mathbf{w}, q) & ,\forall_{q\in L_{2,0}(\Omega)} \\ B' : L_{2,0}(\Omega) \to \mathbf{H}^{-1}(\Omega) & \langle B'r, \mathbf{v}\rangle_{L_2(\Omega)\times L_2(\Omega)} = b(\mathbf{v}, r) & ,\forall_{\mathbf{v}\in\mathbf{H}_0^1(\Omega)} \\ C(\cdot) : \mathbf{H}_0^1(\Omega) \to \mathbf{H}^{-1}(\Omega) & \langle C(\mathbf{w}), \mathbf{v}\rangle_{\mathbf{H}^{-1}(\Omega)\times\mathbf{H}_0^1(\Omega)} = c(\mathbf{w}, \mathbf{w}, \mathbf{v}) & ,\forall_{\mathbf{v}\in\mathbf{H}_0^1(\Omega)}. \end{array}$$

For the numerical treatment of the operator equation (9), we employ a Newton scheme. To this end let us take a closer look at the Fréchet derivative of the operator $\mathcal{F}_{\tau,\nu}(\mathbf{u}, p)$:

For a fixed $\mathbf{u}, \mathbf{v}$ and a small $\mathbf{w}$,

$$c(\mathbf{u} + \mathbf{w}, \mathbf{u} + \mathbf{w}, \mathbf{v}) - c(\mathbf{u}, \mathbf{u}, \mathbf{v}) = c(\mathbf{u}, \mathbf{w}, \mathbf{v}) + c(\mathbf{w}, \mathbf{u}, \mathbf{v}) + o(\mathbf{w}). \tag{8}$$

Thus, the Fréchet derivative $\mathcal{F}_{\tau,\nu}$ in the operator form, is given by

$$D\mathcal{F}_{\tau,\nu}(\mathbf{u}, p) := \begin{pmatrix} A_{\tau,\nu} + N_1(\mathbf{u}) + N_2(\mathbf{u}) & B' \\ B & 0 \end{pmatrix} \tag{9}$$

where for a fixed $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$, the operators $N_1(\mathbf{u}), N_2(\mathbf{u}) : \mathbf{H}_0^1(\Omega) \to \mathbf{H}^{-1}(\Omega)$ are defined by

$$\begin{array}{rl} \langle N_1(\mathbf{u})\mathbf{w}, \mathbf{v}\rangle_{\mathbf{H}^{-1}(\Omega)\times\mathbf{H}_0^1(\Omega)} = c(\mathbf{u}, \mathbf{w}, \mathbf{v}) & ,\forall_{\mathbf{v}\in\mathbf{H}_0^1(\Omega)} \\ \langle N_2(\mathbf{u})\mathbf{w}, \mathbf{v}\rangle_{\mathbf{H}^{-1}(\Omega)\times\mathbf{H}_0^1(\Omega)} = c(\mathbf{w}, \mathbf{u}, \mathbf{v}) & ,\forall_{\mathbf{v}\in\mathbf{H}_0^1(\Omega)}. \end{array}$$

This means that in each step of the Newton scheme, we solve a linear system of the form

$$D\mathcal{F}_{\tau,\nu}(\mathbf{u}, p) \begin{pmatrix} \mathbf{w} \\ r \end{pmatrix} = -\mathcal{F}_{\tau,\nu}(\mathbf{u}, p) \tag{10}$$

and afterwards, update the (approximate) solution by the relation

$$\begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} \leftarrow \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} + \begin{pmatrix} \mathbf{w} \\ r \end{pmatrix}. \tag{11}$$

Consider the nodal bases $\Phi_j := \{\phi_{j,k} : k \in \Lambda_j\}$ and $\Xi_j = \{\xi_{j,k} : k \in \Theta_j\}$ and the ansatz spaces $X_j = \text{span } \Phi_j$ and $M_j = \text{span } \Xi_j$.

To solve numerically the operator equation (7) in the Wavelet-Galerkin sense, we apply the Newton scheme projecting the linear equation (10) onto the approximation spaces $X_j \times M_j$.

Consider the ansatz expansions

$$\mathbf{u}_j = \sum_{k\in\Lambda_j} y_{j,k}\phi_{j,k} := \mathbf{y}_j^T\Phi_j$$
$$\mathbf{w}_j = \sum_{k\in\Lambda_j} x_{j,k}\phi_{j,k} := \mathbf{x}_j^T\Phi_j$$
$$p_j = \sum_{k\in\Theta_j} t_{j,k}\xi_{j,k} := \mathbf{t}_j^T\Xi_j$$
$$r_j = \sum_{k\in\Theta_j} s_{j,k}\xi_{j,k} := \mathbf{s}_j^T\Xi_j$$

and the matrix operators

$$\begin{aligned}
\mathbf{A}_{\tau,\nu;j} &:= (a_{\tau,\nu}(\phi_{j,l},\phi_{j,k}))_{k,l\in\Lambda_j}\\
\mathbf{B}_j &:= (b(\phi_{j,l},\xi_{j,k}))_{k\in\Lambda_j,l\in\Theta_j}\\
\mathbf{N}_{1;j}(\mathbf{y}_j^T\Phi_j) &:= \left(c(\mathbf{y}_j^T\Phi_j,\phi_{j,l},\phi_{j,k})\right)_{k,l\in\Lambda_j}\\
\mathbf{N}_{2;j}(\mathbf{y}_j^T\Phi_j) &:= \left(c(\phi_{j,l},\mathbf{y}_j^T\Phi_j,\phi_{j,k})\right)_{k,l\in\Lambda_j}\\
\mathbf{C}_j(\mathbf{y}_j^T\Phi_j) := \left(c(\mathbf{y}_j^T\Phi_j,\mathbf{y}_j^T\Phi_j,\phi_{j,k})\right)_{ks\in\Lambda_j} &= \mathbf{N}_{2;j}(\mathbf{y}_j^T\Phi_j)\mathbf{y}_j\\
\mathbf{f}_j &:= (\langle\mathbf{f},\phi_{j,l}\rangle)_{l\in\Lambda}
\end{aligned}$$

Now, the Newton scheme (10)-(11) in the nodal basis representation, is given by the equivalent system

$$\mathcal{G}_{\tau,\nu;j}(\mathbf{y}_j)\begin{pmatrix}\mathbf{x}_j\\ \mathbf{t}_j\end{pmatrix} = \mathcal{S}_j(\mathbf{y}_j) \tag{12}$$

$$\mathbf{y}_j \leftarrow \mathbf{x}_j \tag{13}$$

with

$$\mathcal{G}_{\tau,\nu;j}(\mathbf{y}_j) = \begin{pmatrix}\mathbf{A}_{\tau,\nu;j}+\mathbf{N}_{1;j}(\mathbf{y}_j^T\Phi_j)+\mathbf{N}_{2;j}(\mathbf{y}_j^T\Phi_j) & \mathbf{B}_j^T\\ \mathbf{B}_j & \mathbf{0}\end{pmatrix} \tag{14}$$

$$g_j(\mathbf{y}_j) = \begin{pmatrix}\mathbf{N}_{1;j}(\mathbf{y}_j^T\Phi_j)\mathbf{y}_j+\mathbf{f}_j\\ \mathbf{0}\end{pmatrix} \tag{15}$$

Some drawbacks arise in the numerical solution in form of the update of the matrices $\mathbf{N}_{1;j}(\mathbf{y}_j^T\Phi_j), \mathbf{N}_{2;j}(\mathbf{y}_j^T\Phi_j)$ in each iteration, the numerical stability of the scheme and preconditioning strategies to increase the accuracy of the results.

To solve the first drawback, we will use knot oriented quadrature rules to approximate the arising integrals numerically in a problem adapted way. Afterwards, we will prove the convergence of the perturbed system to the original system and moreover, we will introduce some strategies to ensure the stability and to precondition the perturbed system.

3 Multilevel setting

Our goal is to develop Galerkin methods for the approximate solution of (9). However, in difference to conventional finite element discretizations we will work with trial spaces that do not only exhibit the usual approximation properties and good localization but in addition lead to *expansions* of any element in the underlying Hilbert spaces in terms of *multiscale* or *wavelet* bases with certain stability properties. We will show that these stability properties ultimately stated in terms of norm equivalences for Sobolev spaces will indeed allow us to improve on previous theoretical investigations for the above problem. In this section we formulate the relevant facts for our framework. These results are essentially known (cf. [17, 18]) but for the convenience of the reader we include a brief summary of the relevant facts.

3.1 Stable Multiscale Bases

Suppose H is a Hilbert space (of functions defined on Ω, say) with inner product $< \cdot, \cdot >$. Throughout this section orthogonality will always be understood relative to this inner product. Typical examples include $H = L_2(\Omega)$, $H = H^s(\Omega)$ or products of such spaces. Let $\{V_j\}_{j=0}^{\infty}$ be a sequence of closed nested subspaces of H whose union is dense in H. In general the spaces V_j are spanned by *single scale bases* (or *nodal bases*) $\Phi_j = \{\phi_{j,k} : k \in \Lambda_j\}$ which are *uniformly stable*, i.e.,

$$\|c\|_{\ell_2(\Lambda_j)} \sim \| \sum_{k\in\Lambda_j} c_k \phi_{j,k}\|_H \tag{16}$$

uniformly in $j \in \mathbb{N}_0$. Here we denote as usual $\| \cdot \|_H^2 =< \cdot, \cdot >$ and $\|c\|^2_{\ell_2(\Lambda_j)} = \sum_{k\in\Lambda_j} |c_k|^2$.

Successively updating a current approximation in V_{j-1} to a better one in V_j can be facilitated if stable bases

$$\Psi_j = \{\psi_{j,k} : k \in J_j\}$$

for some complement W_j of V_{j-1} in V_j are available. Defining for convenience $\Psi_0 := \Phi_0$, $W_0 := V_0$, any $v_n = \sum_{k\in\Lambda_n} c_k\phi_{n,k} \in V_n$ has then an alternative *multiscale* representation

$$v_n = \sum_{j=0}^{n} \sum_{k\in J_j} d_{j,k}\psi_{j,k}$$

which corresponds to the direct sum decomposition

$$V_n = \bigoplus_{j=0}^{n} W_j.$$

Of course, there is a continuum of possible choices of such complements. Orthogonal decompositions would correspond to wavelets. However, orthogonality often interferes with locality and the actual computation of orthonormal bases might be too expensive. Moreover, in certain applications orthogonal decompositions are actually not best possible [18]. The essential constraint on the choice of W_j is that

$$\Psi = \bigcup_{j \in \mathbb{N}_0} \Psi_j$$

forms a *Riesz-basis* of H, i.e. every $v \in H$ has a unique expansion

$$v = \sum_{j=0}^{\infty} \sum_{k \in J_j} < v, \tilde{\psi}_{j,k} > \psi_{j,k} \tag{17}$$

such that

$$\|v\|_H \sim \left(\sum_{j=0}^{\infty} \sum_{k \in J_j} | < v, \tilde{\psi}_{j,k} > |^2 \right)^{\frac{1}{2}}, \quad v \in H, \tag{18}$$

where $\tilde{\Psi} = \{\tilde{\psi}_{j,k} : k \in J_j, j \in \mathbb{N}_0\}$ forms a biorthogonal system

$$< \psi_{j,k}, \tilde{\psi}_{j',k'} >= \delta_{j,j'} \delta_{k,k'}, \quad j, j' \in \mathbb{N}_0, \quad k \in J_j, \quad k' \in J_{j'} \tag{19}$$

and is in fact also a Riesz-basis for H (cf. [17]).

To explain one aspect why this is important let L_j denote the transformation that takes the coefficients $d_{j,k}$ in the multiscale representation of v_n into the coefficients c_k of the single scale representation. It corresponds to the synthesis part of the fast wavelet transform. In fact, it is known that the Riesz basis property of Ψ is equivalent to L_j and L_j^{-1} being well conditioned, i.e.,

$$\|L_j\|, \|L_j^{-1}\| = \mathcal{O}(1), \quad n \to \infty, \tag{20}$$

where $\| \cdot \|$ denotes the spectral norm [16, 17].

With such a pair of biorthogonal bases Ψ and $\tilde{\Psi}$ one can associate canonical projectors

$$Q_n v := \sum_{j=0}^{n} \sum_{k \in J_j} < v, \tilde{\psi}_{j,k} > \psi_{j,k}, \quad Q_n' v := \sum_{j=0}^{n} \sum_{k \in J_j} < v, \psi_{j,k} > \tilde{\psi}_{j,k}$$

which are obviously adjoints of each other. Of course, when Ψ is a Riesz-basis then the Q_n and hence their adjoints $\tilde{Q}_n$ are uniformly bounded in H. Denoting by $\tilde{V}_n$ the range of Q_n' we have therefore *two* sequences of nested closed subspaces V_j and $\tilde{V}_j$, respectively, whose union is easily seen to be dense in H [16].

While the Riesz-basis property of Ψ implies the existence of a biorthogonal Riesz-basis $\tilde{\Psi}$ as well as the uniform boundedness of the projectors Q_n and Q_n', the converse is known not to be true in general [17]. Additional conditions for verifying the Riesz-basis property for a general Hilbert space setting have been established in [17]. Here we are only interested in their specialization to the particular case $H = L_2(\Omega)$ (where as above Ω is either a closed surface or a domain in $\mathbb{R}^d$) and denote by H^s a corresponding scale of Sobolev spaces. What turns out to matter is that both $\{V_n\}$ and $\{\tilde{V}_n\}$ should have some approximation and regularity properties which can be stated in terms of the following pair of estimates. There exists some $\gamma > 0$ such that the inverse estimate

$$\|v_n\|_{H^s(\Omega)} \lesssim 2^{ns}\|v_n\|_{L_2(\Omega)}, \quad v_n \in V_n, \tag{21}$$

holds for $s < \gamma$. Moreover, there exists some $m \geq \gamma$ such that the direct estimate

$$\inf_{v_n \in V_n} \|v - v_n\|_{L_2(\Omega)} \lesssim 2^{-sn}\|v\|_{H^s(\Omega)}, \quad v \in H^s(\Omega), \tag{22}$$

holds for $s \leq m$. Such estimates are known to hold for every finite element or spline space. For instance, for piecewise linear finite elements one has $\gamma = 3/2, m = 2$.

It will be convenient to introduce the following notation. Let

$$\Lambda := \{\lambda = (j,k) : k \in \Lambda_j, j \in \mathbb{N}_0\} = \bigcup_{j=0}^{\infty}(\{j\} \times \Lambda_j).$$

and define

$$|\lambda| := j \quad \text{if} \quad \lambda \in \Lambda_j.$$

The following result from [17] will play a central role in the subsequent analysis.

Theorem 3.1 *Suppose that $\Psi = \{\psi_\lambda : \lambda \in \Lambda\}$ and $\tilde{\Psi} = \{\tilde{\psi}_\lambda : \lambda \in \Lambda\}$ are biorthogonal collections in $L_2(\Omega)$ and that the associated sequence of projectors $\{Q_j\}_{j=0}^{\infty}$ is uniformly bounded. If both $\{Q_j\}$ and $\{Q'_j\}$ satisfy (21) and (22) relative to some $\gamma, \gamma' > 0$, $\gamma \leq m$, $\gamma' \leq m'$, then*

$$\begin{aligned}
\|v\|_{H^s(\Omega)} \quad &\sim \quad \left(\sum_{\lambda \in \Lambda} 2^{2|\lambda|s}| < v, \tilde{\psi}_\lambda > |^2\right)^{\frac{1}{2}}, \quad s \in (-\gamma', \gamma), \\
&\sim \quad \left(\sum_{\lambda \in \Lambda} 2^{2|\lambda|s}| < v, \psi_\lambda > |^2\right)^{\frac{1}{2}}, \quad s \in (-\gamma, \gamma'), \quad v \in H^s(\Omega).
\end{aligned} \tag{23}$$

Moreover, the projectors Q_j, Q'_j are uniformly bounded in $H^s(\Omega)$, $s \in (-\gamma', \gamma)$ and $s \in (-\gamma, \gamma')$, respectively.

For more information about the construction of multiscale bases $\Psi, \tilde{\Psi}$ with the above properties the reader is referred to [22]. Throughout the remainder of this paper we will assume that Ψ and $\tilde{\Psi}$ satisfy the assumptions of Theorem 3.1 and for

$$\gamma > t + \frac{d}{2}, \quad \gamma' > -t + \frac{d}{2}, \tag{24}$$

we know that there exist positive constants $0 < c_3, c_4 < \infty$ for which

$$c_3 \left(\sum_{\lambda \in \Lambda} 2^{-2t|\lambda|} | < v, \psi_\lambda > |^2 \right)^{1/2} \leq \|v\|_{-t} \leq c_4 \left(\sum_{\lambda \in \Lambda} 2^{-2t|\lambda|} | < v, \psi_\lambda > |^2 \right)^{1/2} . \tag{25}$$

Finally, for our applications it will be important to work with *local bases*, i.e., we will always assume that

$$diam\,(supp\phi_{n,k})\,, \quad diam(supp\psi_{n,k}) \sim 2^{-n}, \quad n \in \mathbb{N}. \tag{26}$$

Furthermore, it is desirable that the $\tilde{\phi}_{n,k}$, $\tilde{\psi}_{n,k}$ have the same property

$$diam(supp\tilde{\phi}_{n,k}), \quad diam(supp\tilde{\psi}_{n,k}) \sim 2^{-n}, \quad n \in \mathbb{N}. \tag{27}$$

3.2 Interpolation Projectors and knot oriented quadrature rules

In our approach we propose to approximate the trilinear form $c(\mathbf{z}, \mathbf{w}, \mathbf{v})$ by a suitable quadrature rule.

For this purpose it is convenient to work with *interpolating* refinable functions, i.e., one requires that ϕ is at least continuous and satisfies the interpolation property

$$\phi(k) = \delta_{0,k}, \qquad k \in \mathbb{Z}^d. \tag{28}$$

and the integral property

$$\int_{\mathbb{R}^d} \phi(x)dx = 1 \tag{29}$$

As already stated, we only consider compactly supported scaling functions. Furthermore, functions ϕ which are sufficiently smooth and well-located are preferable. In recent studies, several examples of refinable functions satisfying these conditions have been constructed, see, e.g., [13, 23, 24, 25, 26, 31]. In particular, in all our numerical computations, we select the interpolating scaling functions from the family of the so-called Deslauriers-Dubuc *fundamental functions*. These functions, which are obtained via auto-correlation of the well-known compactly supported orthogonal Daubechies scaling functions, have very attractive properties; see, e.g., [25, 26]. In fact, if we denote by $\phi := \phi^{2N}$ the Deslauriers-Dubuc scaling function of order $2N$, obtained as autocorrelation of the Daubechies scaling function associated to the parameter N, we have that ϕ has compact support and is interpolating. Moreover, ϕ^{2N} has polynomial exactness $2N - 1$ and its smoothness increases with N.

An algorithm for constructing a dual scaling function $\tilde{\phi}$ for a given interpolating scaling function ϕ was developed in [28].

Let us restate some basic facts about interpolation projector related to ϕ:

For what follows, we will denote by $\phi := (\phi^1, \phi^2, \dots, \phi^d)$ the vector field of the interpolating scaling functions such that $\phi^m = \phi^n$. To solve a linear system in each Newton iteration, our main goal is consists in approximate the trilinear form $c(\mathbf{z}_j, \mathbf{w}_j, \mathbf{v}_j)$ by a suitable quadrature rule. In our case, we consider for each component of $\phi := (\phi^1, \phi^2, \ldots, \phi^d)$ and for a real valued continuous function f which satisfy periodic conditions, the interpolating projectors I_j^n defined by

$$I_j^n(f) = \sum_{k \in \Lambda_j} f(2^{-j}k)\phi^n(2^j \cdot -k) \tag{30}$$

and the quadrature rule

$$\pi_j^n(f) = \int_\Omega I_j^n(f)dx \tag{31}$$

We state a first result on the operators I_j^n (c.f. [10]).

Lemma 3.1 *The operator I_j^n is a bounded operator from L_∞ to L_∞.*

As a consequence, we obtain the following theorem (again, c.f. [10]).

Theorem 3.2 *Let f be uniformly Hölder-α. Then, we have*

$$||f - I_j^n f||_\infty \leq C2^{-jd\alpha}.$$

For what follows, let

$$\tilde{c}_j(\mathbf{z}, \mathbf{w}, \mathbf{v}) = \sum_{m,n=1}^{d} \pi_j^n \left(z^m \frac{\partial w^n}{\partial x_m} v^n \right)$$

be the resulting perturbed trilinear form.

Using the interpolating property (28) and the integral property (29) our quadrature rule (31) becomes then

$$\begin{aligned}
\pi_j^n(f) &= \int_\Omega I_j^n(f)dx = \sum_{l \in \Lambda_j} f(2^{-j}l) \int_\Omega \phi^n(2^j x - l)dx \\
&= \sum_{l \in \Lambda_j} f(2^{-j}l) \left(2^{-jd} \int_\Omega \phi^n(x)dx \right) \\
&= 2^{-jd} \sum_{l \in \Lambda_j} f(2^{-j}l). \qquad (32)
\end{aligned}$$

Applying (32) to $\tilde{c}_j(\mathbf{z}, \mathbf{w}, \mathbf{v})$, we obtain

$$\tilde{c}_j(\mathbf{z}, \mathbf{w}, \mathbf{v}) = 2^{-jd} \sum_{m,n=1}^{d} z^m(2^{-j}k) \frac{\partial w^n}{\partial x_m}(2^{-j}k) v^n(2^{-j}k) \tag{33}$$

4 Applications to the Navier-Stokes equations

4.1 Construction of discretized problem

According to our approach, we solve numerically a perturbed version of the system (12), where we consider

$$\tilde{\mathcal{G}}_{\tau,\nu;j}(\mathbf{y}_j) = \begin{pmatrix} \mathbf{A}_{\tau,\nu;j} + \tilde{\mathbf{N}}_{1;j}(\mathbf{y}_j^T \Phi_j) + \tilde{\mathbf{N}}_{2;j}(\mathbf{y}_j^T \Phi_j) & \mathbf{B}_j^T \\ \mathbf{B}_j & \mathbf{0} \end{pmatrix} \tag{34}$$

$$\tilde{g}_j(\mathbf{y}_j) = \begin{pmatrix} \tilde{\mathbf{N}}_{1;j}(\mathbf{y}_j^T \Phi_j)\mathbf{y}_j + \mathbf{f}_j \\ \mathbf{0} \end{pmatrix} \tag{35}$$

instead of $\mathcal{G}_{\tau,\nu;j}(\mathbf{y}_j)$ and $g_j(\mathbf{y}_j)$, respectively.

Here $\mathbf{A}_{\tau,\nu;j}$, $\tilde{\mathbf{N}}_{1;j}(\mathbf{y}_j^T \Phi_j)$ and $\tilde{\mathbf{N}}_{2;j}(\mathbf{y}_j^T \Phi_j)$ are diagonal block matrices with diagonal blocks $\mathbf{A}^n_{\tau,\nu;j}$,$\tilde{\mathbf{N}}^n_{1;j}(\mathbf{y}_j^T \Phi_j)$, and $\tilde{\mathbf{N}}^n_{2;j}(\mathbf{y}_j^T \Phi_j)$, for $n = 1, \dots, d$. $\mathbf{B}_j$ is a matrix with column blocks $\mathbf{B}_j^n$ and $\mathbf{f}_j$ is a column vector with row blocks $\mathbf{f}_j^n$ for $n = 1, \dots, d$.

Details for the construction of the matrices $\mathbf{A}_{\tau,\nu;j}$ and $\mathbf{B}_j$ can be found in, e.g., [20]. According to (3.2), for sufficiently smooth functions, we can construct each row block $\mathbf{f}_j^n$ by using the formula (32).

For the construction of the perturbed matrices $\tilde{\mathbf{N}}_{p;j}(\mathbf{y}_j^T \Phi_j)$, $p = 1, 2$, we take into account the perturbed trilinear form $\tilde{c}_j(\mathbf{z}, \mathbf{w}, \mathbf{v})$ obtained in (33).

Using (12) and the interpolating property (28) for each component of the scaling function ϕ, the entries of the matrix blocks $\tilde{\mathbf{N}}^n_{1;j}(\mathbf{y}_j^T \Phi_j)$ and $\tilde{\mathbf{N}}^n_{2;j}(\mathbf{y}_j^T \Phi_j)$ are given by

$$\left(\tilde{\mathbf{N}}^n_{2;j}(\mathbf{y}_j^T \Phi_j)\right)_{k,l\in\Lambda_j} = \delta_{k,l} \cdot \operatorname{grad}\left(\mathbf{y}_j^{n\,T} \Phi_j^n\right)(2^{-j}k)$$

and

$$\left(\tilde{\mathbf{N}}^n_{2;j}(\mathbf{y}_j^T \Phi_j)\right)_{k,l\in\Lambda_j} = \sum_{m=1}^{d} \frac{\partial \phi^n_{j,l}}{\partial x_m}(2^{-j}k) y^n_{j,k},$$

respectively, for each $n = 1, \dots, d$. Hereby, $\mathbf{y}_j := (\mathbf{y}_j^1, \dots, \mathbf{y}_j^d)^T$.

In both cases, the construction of the matrix blocks $\tilde{\mathbf{N}}^n_{p;j}(\mathbf{y}_j^T \Phi_j), p = 1, 2$, involves the evaluation of the partial derivatives of the scaling functions.

Fixing $M_j^{m,n} := \left(\frac{\partial \phi^n_{j,l}}{\partial x_m}(2^{-j}k)\right)_{k,l\in\Lambda_j}$ and expanding $\frac{\partial \phi^n_{j,l}}{\partial x_m}(2^{-j}k)$ in a biorthogonal expansion, we obtain due the interpolating property (28) the identity

$$\frac{\partial \phi^n_{j,l}}{\partial x_m}(2^{-j}k) = 2^{jd/2} \left\langle \frac{\partial \phi^n_{j,l_m}}{\partial x_m}, \tilde{\phi}^n_{j,k_m} \right\rangle$$

and, moreover,

$$\left(\operatorname{grad}\left(\mathbf{y}_j^{n\,T} \Phi_j^n\right)(2^{-j}k)\right)_{k\in\Lambda_j} = \sum_{m=1}^{d} M_j^{m,n} \mathbf{y}_j^n.$$

Consequently, we obtain for each $n = 1, \ldots, d$, the formulae

$$\tilde{N}^n_{1;j}(\mathbf{y}_j^T \Phi_j) = \mathrm{diag}(\mathbf{y}_j^n) \left(\sum_{m=1}^{d} M_j^{m,n} \right) \tag{36}$$

$$\tilde{N}^n_{2;j}(\mathbf{y}_j^T \Phi_j) := 2^{-jd/2} \,\mathrm{diag} \left(\sum_{m=1}^{d} M_j^{m,n} \mathbf{y}_j^n \right), \tag{37}$$

where $\mathrm{diag}(x)$ denotes the diagonal matrix associated to the column vector x.

In the next section, we will prove that the exact solution of the problem (7) in $X_j \times M_j$ is the limit of the solution of the perturbed problem

$$\tilde{\mathcal{F}}_{\tau,\nu;j}(\mathbf{u}, p) := \begin{pmatrix} A_{\tau,\nu} & B' \\ B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} + \begin{pmatrix} \tilde{C}_j(\mathbf{u}) - \mathbf{f} \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 0 \end{pmatrix} \tag{38}$$

in $X_j \times M_j$, where $\tilde{C}_j$ is a nonlinear operator defined by

$$\langle \tilde{C}_j(\mathbf{u}), \mathbf{v} \rangle_{\mathbf{H}^{-1}(\Omega) \times \mathbf{H}^1_0(\Omega)} = \tilde{c}_j(\mathbf{u}, \mathbf{u}, \mathbf{v}) \quad \forall_{\mathbf{v} \in \mathbf{H}^1_0(\Omega)}$$

4.2 Convergence Result

In this section, we consider the convergence of our Galerkin approximation of the Navier-Stokes equation (1)-(4). Hence, we aim to estimate the error

$$\|\mathbf{u} - \mathbf{u}_j^*\|_{\mathbf{H}^1(\Omega)} + \|p - p_j^*\|_{L_2(\Omega)},$$

where $(\mathbf{u}_j^*, p_j^*)$ is assumed to be the limit of the Newton scheme. This can be done by means of the following decomposition

$$\|\mathbf{u} - \mathbf{u}_j^*\|_{\mathbf{H}^1(\Omega)} \leq \|\mathbf{u} - \mathbf{u}_j\|_{\mathbf{H}^1(\Omega)} + \|\mathbf{u}_j - \tilde{\mathbf{u}}_j\|_{\mathbf{H}^1(\Omega)} + \|\tilde{\mathbf{u}}_j - \mathbf{u}_j^*\|_{\mathbf{H}^1(\Omega)},$$

$$\|p - p_j^*\|_{L_2(\Omega)} \leq \|p - p_j\|_{L_2(\Omega)} + \|p_j - \tilde{p}_j\|_{L_2(\Omega)} + \|\tilde{p}_j - p_j^*\|_{L_2(\Omega)}$$

where $(\mathbf{u}_j, p_j)$ is the exact solution of (7) in $X_j \times M_j$, $(\tilde{\mathbf{u}}_j, \tilde{p}_j)$ is the solution of the perturbed system (38) in $X_j \times M_j$ and $(\mathbf{u}_j^*, p_j^*)$ is the Newton approximation of $(\tilde{\mathbf{u}}_j, \tilde{p}_j)$.

For the Newton approximation we have quadratic convergence for a sufficiently close initial approximation. The terms $\|\mathbf{u} - \mathbf{u}_j\|_{\mathbf{H}^1(\Omega)}$ and $\|p - p_j\|_{L_2(\Omega)}$ can be estimated by the classical abstract estimates and the Jackson-type theorems for our wavelet bases, for instance, see [30], [9], or [10]. Therefore, it is enough to prove the following theorem.

Theorem 4.1 *Let $(\mathbf{u}_j, p_j)$ be the exact solution on the space $X_j \times M_j$ system (7), $(\tilde{\mathbf{u}}_j, \tilde{p}_j)$ the exact solution of the perturbed system $X_j \times M_j$. If for each $\mathbf{v} \in X_j$, the Fréchet derivative $A_{\tau,\nu} + \tilde{C}_j(\mathbf{v})$ is invertible on* $\ker B$ *and if for $\alpha > 0$, $(\mathbf{u}_j \cdot \mathrm{grad})\mathbf{u}_j$ is in the Sobolev space $H^s(\Omega)$, $s > d/2 + \alpha$, then there exists positive constants $\alpha_j, \tilde{\alpha}_j, \tilde{\beta}_j$ and k_1 and such that*

$$\|\mathbf{u}_j - \tilde{\mathbf{u}}_j\|_{\mathbf{H}_0^1(\Omega)} \leq \frac{1}{\tilde{\alpha}_j} k_1 2^{-dj\alpha} \tag{39}$$

$$\|p_j - \tilde{p}_j\|_{L_2(\Omega)} \leq \frac{1}{\tilde{\beta}_j}\left(1 + \frac{\alpha_j}{\tilde{\alpha}_j}\right) k_1 2^{-dj\alpha} \tag{40}$$

Proof: Consider the Navier-Stokes problem in the operator form (9),

$$\begin{aligned} A_{\tau,\nu}\mathbf{u}_j + B' p_j + C(\mathbf{u}_j) &= \mathbf{f} \\ B\mathbf{u}_j &= 0 \end{aligned}$$

and its associated problem arising by applying the quadrature rule

$$\begin{aligned} A_{\tau,\nu}\tilde{\mathbf{u}}_j + B' \tilde{p}_j + \tilde{C}_j(\tilde{\mathbf{u}}_j) &= \mathbf{f} \\ B\tilde{\mathbf{u}}_j &= 0 \end{aligned}$$

Denoting by $\mathbf{v}_j = \mathbf{u}_j - \tilde{\mathbf{u}}_j$ and $q_j = p_j - \tilde{p}_j$, we obtain

$$\begin{aligned} A_{\tau,\nu}\mathbf{v}_j + C(\mathbf{u}_j) - \tilde{C}_j(\tilde{\mathbf{u}}_j) + B' q_j &= \mathbf{0} \\ B\mathbf{v}_j &= 0 \end{aligned}$$

that is,

$$A_{\tau,\nu}\mathbf{v}_j + \tilde{C}_j(\mathbf{u}_j) - \tilde{C}_j(\tilde{\mathbf{u}}_j) + B' q_j = \tilde{C}_j(\mathbf{u}_j) - C(\mathbf{u}_j) \tag{41}$$

$$B\mathbf{v}_j = 0 \tag{42}$$

At this stage, we assume for each $\mathbf{v} \in \mathbf{X}_j$ that the Fréchet derivative of the operator $A_{\tau,\nu} + \tilde{C}_j(\mathbf{v})$ is invertible on $\ker B$.

Looking at the perturbed operator $\tilde{C}_j$ in the *mixed* form and choosing $\mathbf{w}_j \in X_j$ sufficiently small, as in (8), we have the relation

$$\langle \tilde{C}_j(\mathbf{u}_j) - \tilde{C}_j(\tilde{\mathbf{u}}_j), \mathbf{w}_j \rangle_{\mathbf{H}^{-1}(\Omega)\times\mathbf{H}_0^1(\Omega)} = \tilde{c}_j(\tilde{\mathbf{u}}_j, \mathbf{v}_j, \mathbf{w}_j) + \tilde{c}_j(\mathbf{v}_j, \tilde{\mathbf{u}}_j, \mathbf{v}_j) + o(\mathbf{w}_j).$$

Therefore,

$$\langle \tilde{C}_j(\mathbf{u}_j) - \tilde{C}_j(\tilde{\mathbf{u}}_j), \mathbf{w}_j \rangle_{\mathbf{H}^{-1}(\Omega)\times\mathbf{H}_0^1(\Omega)} = \langle (\tilde{N}_{1;j}(\tilde{\mathbf{u}}_j) + \tilde{N}_{2;j}(\tilde{\mathbf{u}}_j))\mathbf{v}_j, \mathbf{w}_j \rangle_{\mathbf{H}^{-1}(\Omega)\times\mathbf{H}_0^1(\Omega)}, \tag{43}$$

where $\tilde{N}_{1;j}(\tilde{\mathbf{u}}_j) + \tilde{N}_{2;j}(\tilde{\mathbf{u}}_j)$ is the Fréchet derivative of the perturbed operator $\tilde{C}_j(\tilde{\mathbf{u}}_j)$.

Moreover, as in [5], page 133, we have the estimates

$$||\mathbf{v}_j||_{\mathbf{H}_0^1(\Omega)} \leq \frac{1}{\tilde{\alpha}_j} \|\tilde{C}_j(\mathbf{u}_j) - C(\mathbf{u}_j)\|_{\mathbf{H}^{-1}(\Omega)} \tag{44}$$

$$||q_j||_{L_2(\Omega)} \leq \frac{1}{\tilde{\beta}_j}\left(1 + \frac{\alpha_j}{\tilde{\alpha}_j}\right) \|\tilde{C}_j(\mathbf{u}_j) - C(\mathbf{u}_j)\|_{\mathbf{H}^{-1}(\Omega)}, \tag{45}$$

where the constants $0 < \tilde{\alpha}_j \leq \alpha_j < \infty$ comes from the norm equivalence

$$\tilde{\alpha}_j \|\mathbf{v}_j\|_{\mathbf{H}_0^1(\Omega)} \leq \|A_{\tau,\nu}\mathbf{v}_j + \tilde{N}_{1;j}(\tilde{\mathbf{u}}_j)\mathbf{v}_j + \tilde{N}_{2;j}(\tilde{\mathbf{u}}_j)\mathbf{v}_j\|_{\mathbf{H}^{-1}(\Omega)} \leq \alpha_j \|\mathbf{v}_j\|_{\mathbf{H}_0^1(\Omega)}$$

and $\tilde{\beta}_j > 0$ comes from the inf-sup (or LBB) condition

$$\inf_{q \in M_j} \sup_{\mathbf{v} \in X_j} \frac{\langle B\mathbf{v}, q\rangle_{\mathbf{H}^{-1}(\Omega)\times \mathbf{H}_0^1(\Omega)}}{\|\mathbf{v}\|_{\mathbf{H}_0^1(\Omega)} \|\mathbf{v}\|_{\mathbf{H}_0^1(\Omega)}} \geq \tilde{\beta}_j, \tag{46}$$

On the other hand, under the assumption that $(\mathbf{u}_j \cdot \mathrm{grad})\mathbf{u}_j$ is in the Sobolev space $H^s(\Omega)$, $s > d/2 + \alpha$, by Sobolev's embedding theorem [1] we conclude that $(\mathbf{u}_j \cdot \mathrm{grad})\mathbf{u}_j$ is in the Hölder space $C^\alpha(\Omega)$. As a consequence of Theorem 3.2, we obtain the estimate

$$|\tilde{c}_j(\mathbf{u}_j, \mathbf{u}_j, \mathbf{w}_j) - c(\mathbf{u}_j, \mathbf{u}_j, \mathbf{w}_j)| \leq k_1 2^{-dj\alpha} \tag{47}$$

for some positive constant k_1 independent of j.

Therefore, we obtain the upper estimates (39) and (40) for $\|\mathbf{u}_j - \tilde{\mathbf{u}}_j\|_{\mathbf{H}^1(\Omega)}$ and $\|p_j - \tilde{p}_j\|_{\mathbf{H}^1(\Omega)}$, respectively. ∎

4.3 Newton based steepest descendent scheme

The implementation of the Newton scheme involves, in each iteration, the resolution of a saddle point problem. Instead of considering the nodal bases Φ_j and Ξ_j, to span X_j and M_j, respectively, we will consider the multiscale bases $\Psi := \bigcup_{i=0}^{j} \Psi_i$ and $\Upsilon := \bigcup_{i=0}^{j} \Upsilon_i$, where Ψ_i and Υ_i are stable bases for the wavelet spaces $X_i \ominus X_{i-1}$ and $M_i \ominus M_{i-1}$, respectively.

As it was shown in [17, 19, 29], working in the multiscale sense allows us to construct preconditioners in an easier way as compared with the construction of the preconditioners in the multigrid and finite elements sense.

Let $\mathbf{L}_j$ denote the matrix which transforms the coefficients relative to Ψ_j into those relative to the nodal basis Φ_j and $\mathbf{U}_j$ denote the matrix which transforms the coefficients relative to Υ_j into those relative to the nodal basis Ξ_j.

Choosing

$$\mathcal{Q}_j := \begin{pmatrix} \mathbf{L}_j^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_j^{-1} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{x}_\Psi \\ \mathbf{t}_\Psi \end{pmatrix} := \mathcal{Q}_j \begin{pmatrix} \mathbf{x}_j \\ \mathbf{t}_j \end{pmatrix} = \begin{pmatrix} \mathbf{L}_j^{-1}\mathbf{x}_j \\ \mathbf{U}_j^{-1}\mathbf{s}_j \end{pmatrix}$$

$$\mathbf{y}_\Psi = \mathbf{L}_j^{-1}\mathbf{y}_j$$

and considering the multi-scale representations for the matrix operators

$$\begin{aligned} \mathbf{A}_{\tau,\nu;\Psi} &:= \mathbf{L}_j^{-1}\mathbf{A}_{\tau,\nu;j}\mathbf{L}_j; \\ \tilde{\mathbf{N}}_{n;\Psi}(\mathbf{y}_j^T\Phi_j) &:= \mathbf{L}_j^{-1}\tilde{\mathbf{N}}_{n;j}(\mathbf{y}_j^T\Phi_j)\mathbf{L}_j, \qquad n = 1, 2 \\ \mathbf{f}_\Psi &:= \mathbf{L}_j^{-1}\mathbf{f}_j, \\ \mathbf{B}_{\Psi,\Upsilon} &:= \mathbf{U}_j^{-1}\mathbf{B}_j\mathbf{L}_j \end{aligned}$$

the equation in the multilevel basis is given by

$$\tilde{\mathcal{G}}_{\tau,\nu;\Psi,\Upsilon}(\mathbf{y}_j)\begin{pmatrix}\mathbf{x}_\Psi\\ \mathbf{t}_\Psi\end{pmatrix} = \tilde{g}_\Psi(\mathbf{y}_j),$$

with

$$\tilde{\mathcal{G}}_{\tau,\nu;\Psi,\Upsilon}(\mathbf{y}_j) := \mathcal{Q}_j\tilde{\mathcal{G}}_{\tau,\nu;j}(\mathbf{y}_j) = \begin{pmatrix}\mathbf{A}_{\tau,\nu;\Psi} + \tilde{\mathbf{N}}_{1;\Psi}(\mathbf{y}_j^T\Phi_j) + \tilde{\mathbf{N}}_{2;\Psi}(\mathbf{y}_j^T\Phi_j) & \mathbf{B}_{\Psi,\Upsilon}^T\\ \mathbf{B}_{\Psi,\Upsilon} & \mathbf{0},\end{pmatrix} \tag{48}$$

$$\tilde{g}_\Psi(\mathbf{y}_j) := \mathcal{Q}_j\tilde{g}_j(\mathbf{y}_j) = \begin{pmatrix}\tilde{\mathbf{N}}_{1;\Psi}(\mathbf{y}_j^T\Phi_j)\mathbf{L}_j^{-1}\mathbf{y}_\Psi + \mathbf{f}_\Psi\\ \mathbf{0}\end{pmatrix}. \tag{49}$$

There are a lot of approaches concerning the numerical solution of saddle point problems, see e.g. [3, 4, 6, 7, 11, 14, 21, 29]. Two of the most popular approaches implemented in the wavelet context are the Uzawa scheme and the preconditioning conjugate gradient scheme proposed by Bramble and Pasciak in [6] and further developed by Kunoth [29] in the wavelet and multiscale sense.

One of the drawbacks of Uzawa algorithm is that the convergence of the scheme depends on the spectra of the Schur complement, and hence, we cannot precondition the Schur complement associated to the $(1,1)$-block without computing its inverse. On the contrary, the Bramble and Pasciak approach allows us to preconditioning the system without computing the inverse of the $(1,1)$-block. Taking up this approach, one can turn a saddle point problem into a positive definite one. It requires a good preconditioner for the $(1,1)$-block in (48).

For questions concerning the preconditioning of the saddle point system, we will adopt a steepest descendent variant of the Bramble and Pasciak approach, [6]. Let us take a close look to this approach:

Let

$$\tilde{\mathbf{F}}_{\tau,\nu;\Psi}(\mathbf{y}_j) = \mathbf{A}_{\tau,\nu;\Psi} + \tilde{\mathbf{N}}_{1;\Psi}(\mathbf{y}_j^T\Phi_j) + \tilde{\mathbf{N}}_{2;\Psi}(\mathbf{y}_j^T\Phi_j).$$

Suppose that $\tilde{\mathbf{P}}_{\tau,\nu;\Psi}(\mathbf{y}_j)$ is a positive definite preconditioner of $\tilde{\mathbf{F}}_{\tau,\nu;\Psi_j}(\mathbf{y}_j)$, that is

$$\gamma_j\mathbf{v}^T\tilde{\mathbf{P}}_{\tau,\nu;\Psi}(\mathbf{y}_j)\mathbf{v} \le \mathbf{v}^T\tilde{\mathbf{F}}_{\tau,\nu;\Psi}(\mathbf{y}_j)\mathbf{v} \le \Gamma_j\mathbf{v}^T\tilde{\mathbf{P}}_{\tau,\nu;\Psi}(\mathbf{y}_j)\mathbf{v} \tag{50}$$

holds for all $\mathbf{v} \in \ell_2(\Lambda_j)$ with $1 < \gamma_j \le \Gamma_j < \infty$. Thus,

$$(\gamma_j-1)\mathbf{v}^T\tilde{\mathbf{P}}_{\tau,\nu;\Psi}(\mathbf{y}_j)\mathbf{v} \le \mathbf{v}^T\left(\tilde{\mathbf{F}}_{\tau,\nu;\Psi}(\mathbf{y}_j) - \tilde{\mathbf{P}}_{\tau,\nu;\Psi}(\mathbf{y}_j)\right)\mathbf{v} \le \left(1-\frac{1}{\Gamma_j}\right)\mathbf{v}^T\tilde{\mathbf{F}}_{\tau,\nu;\Psi}(\mathbf{y}_j)\mathbf{v}$$

holds for all $\mathbf{v} \neq 0$.

Moreover, the sesquilinear form

$$\left[\begin{pmatrix}\mathbf{u}\\ \mathbf{p}\end{pmatrix},\begin{pmatrix}\mathbf{v}\\ \mathbf{q}\end{pmatrix}\right] := \mathbf{u}^T\left(\tilde{\mathbf{F}}_{\tau,\nu;\Psi}(\mathbf{y}_j) - \tilde{\mathbf{P}}_{\tau,\nu;\Psi}(\mathbf{y}_j)\right)\mathbf{v} + \mathbf{p}^T\mathbf{q}$$

is positive definite.

Pre-multiplication of the matrix $\tilde{\mathcal{G}}_{\tau,\nu;\Psi}(\mathbf{y}_j)$ and the right hand side $\tilde{g}_\Psi(\mathbf{y}_j)$ by the matrix

$$\mathcal{M}_{\Psi,\Upsilon}(\mathbf{y}_j) = \begin{pmatrix} \tilde{\mathbf{P}}_{\tau,\nu;\Psi}(\mathbf{y}_j)^{-1} & \mathbf{0} \\ \mathbf{B}_{\Psi,\Upsilon}\tilde{\mathbf{P}}_{\tau,\nu;\Psi}(\mathbf{y}_j)^{-1} & -\mathbf{I} \end{pmatrix}$$

yields the system

$$\hat{\mathcal{G}}_{\tau,\nu;\Psi,\Upsilon}(\mathbf{y}_j) \begin{pmatrix} \mathbf{x}_\Psi \\ \mathbf{t}_\Psi \end{pmatrix} = \hat{g}_{\Psi,\Upsilon}(\mathbf{y}_j) \tag{51}$$

with

$$\hat{\mathcal{G}}_{\tau,\nu;\Psi,\Upsilon}(\mathbf{y}_j) = \begin{pmatrix} \tilde{\mathbf{P}}_{\tau,\nu;\Psi}(\mathbf{y}_j)^{-1}\tilde{\mathbf{F}}_{\tau,\nu;\Psi}(\mathbf{y}_j) & \tilde{\mathbf{P}}_{\tau,\nu;\Psi}(\mathbf{y}_j)^{-1}\mathbf{B}^T_{\Psi,\Upsilon} \\ \mathbf{B}_{\Psi,\Upsilon}\left(\tilde{\mathbf{P}}_{\tau,\nu;\Psi}(\mathbf{y}_j)^{-1}\tilde{\mathbf{F}}_{\tau,\nu;\Psi}(\mathbf{y}_j) - \mathbf{I}\right) & \mathbf{B}_{\Psi,\Upsilon}\tilde{\mathbf{P}}_{\tau,\nu;\Psi}(\mathbf{y}_j)^{-1}\mathbf{B}^T_{\Psi,\Upsilon} \end{pmatrix} \tag{52}$$

$$\hat{g}_{\Psi,\Upsilon}(\mathbf{y}_j) = \begin{pmatrix} \tilde{\mathbf{P}}_{\tau,\nu;\Psi}(\mathbf{y}_j)^{-1}\left(\tilde{\mathbf{N}}_{1;\Psi}(\mathbf{y}_j^T\Phi_j)\mathbf{L}_j^{-1}\mathbf{y}_\Psi + \mathbf{f}_\Psi\right) \\ \mathbf{B}_{\Psi,\Upsilon}\tilde{\mathbf{P}}_{\tau,\nu;\Psi}(\mathbf{y}_j)^{-1}\left(\tilde{\mathbf{N}}_{1;\Psi}(\mathbf{y}_j)\mathbf{L}_j^{-1}\mathbf{y}_\Psi + \mathbf{f}_\Psi\right) \end{pmatrix}. \tag{53}$$

Let us remark that the matrix $\hat{\mathcal{G}}_{\tau,\nu;\Psi,\Upsilon}(\mathbf{y}_j)$ is positive definite relative to the sesquilinear form $[\cdot,\cdot]$.

Using this fact, we can implement as in [6] a steepest descendent algorithm to solve the system (51) relative to the sesquilinear form $[\cdot,\cdot]$.

Furthermore, according to [6, 29], the matrix operator

$$\hat{\mathcal{P}}_{\tau,\nu;\Psi,\Upsilon}(\mathbf{y}_j) := \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{\Psi,\Upsilon}\tilde{\mathbf{F}}_{\tau,\nu;\Psi}(\mathbf{y}_j)^{-1}\mathbf{B}^T_{\Psi,\Upsilon} \end{pmatrix}$$

is spectrally equivalent to the matrix operator $\hat{\mathcal{G}}_{\tau,\nu;\Psi,\Upsilon}(\mathbf{y}_j)$ relative to the functional $[\cdot,\cdot]$, in the sense that

$$\begin{aligned} \mu_j \left[\hat{\mathcal{P}}_{\tau,\nu;\Psi,\Upsilon}(\mathbf{y}_j)\begin{pmatrix}\mathbf{v}\\ \mathbf{q}\end{pmatrix}, \begin{pmatrix}\mathbf{v}\\ \mathbf{q}\end{pmatrix}\right] &\le \left[\hat{\mathcal{G}}_{\tau,\nu;\Psi}(\mathbf{y}_j)\begin{pmatrix}\mathbf{v}\\ \mathbf{q}\end{pmatrix}, \begin{pmatrix}\mathbf{v}\\ \mathbf{q}\end{pmatrix}\right] \\ &\le \sigma_j \left[\hat{\mathcal{P}}_{\tau,\nu;\Psi,\Upsilon}(\mathbf{y}_j)\begin{pmatrix}\mathbf{v}\\ \mathbf{q}\end{pmatrix}, \begin{pmatrix}\mathbf{v}\\ \mathbf{q}\end{pmatrix}\right] \end{aligned}$$

holds for all $(\mathbf{v},\mathbf{q}) \in \ell_2(\Lambda_j) \times \ell_2(\Theta_j)$, where

$$\mu_j = \left(\frac{3\Gamma_j - 1}{2\Gamma_j} + \sqrt{\left(1 - \frac{1}{\Gamma_j}\right) + \frac{\left(1 - \frac{1}{\Gamma_j}\right)^2}{4}}\right)^{-1}$$

$$\sigma_j = \Gamma_j\left(1 + \sqrt{1 - \frac{1}{\Gamma_j}}\right)$$

The constants μ_j and σ_j tend uniformly to one as Γ_j tends to one, implying that the eigenvalues of $\hat{\mathcal{P}}_{\tau,\nu;\Psi}(\mathbf{y}_j)^{-1}\hat{\mathcal{G}}_{\tau,\nu;\Psi}(\mathbf{y}_j)$ tend to one uniformly. Therefore, $\tilde{\mathbf{P}}_{\tau,\nu;\Psi}(\mathbf{y}_j)$ should be scaled such that (50) holds with Γ_j not so far from one. Furthermore, the constants indicate that the spectral condition number of $\hat{\mathcal{P}}_{\tau,\nu;\Psi}(\mathbf{y}_j)^{-1}\hat{\mathcal{G}}_{\tau,\nu;\Psi}(\mathbf{y}_j)$ with respect to $[\cdot,\cdot]$ grows, at most, proportional to the largest eigenvalue of $\tilde{\mathbf{P}}_{\tau,\nu;\Psi}(\mathbf{y}_j)^{-1}\tilde{\mathbf{F}}_{\tau,\nu;\Psi}(\mathbf{y}_j)$.

Now our main problem is reduced to find a preconditioner for the matrix $\tilde{\mathbf{F}}_{\tau,\nu;\Psi}(\mathbf{y}_j)$. Contrary to the linear case presented in [19] and [29], there are no well-establish strategies to construct wavelet preconditioners for the nonlinear case. In the nonlinear case, we need to construct a preconditioner in each Newton step which depends on the level j and on the last approximation.

We will develop an approach in this direction in the section 5.

5 Wavelet Preconditioning

Typically, stiffness-type matrices in the nodal basis exhibit a polynomial growth rate of spectral condition proportional to the size. There is a whole theory concerning preconditioning strategies, see e.g., [19, 20, 29]. Concerning the $\mathbf{H}^1$-coercivity of the operator it can be easily shown, under the stability of the wavelet basis, so that, our scheme would converge rapidly. In the non-coercive case, things are slightly complicated.

We assume that our solution is regular in each iteration and for each level j, that is

$$\sup_{\mathbf{v}\in X_j} \frac{a_{\tau,\nu}(\mathbf{v},\mathbf{v}) + \tilde{c}_j(\mathbf{v},\mathbf{y}_j^T\Phi_j,\mathbf{v}) + \tilde{c}_j(\mathbf{y}_j^T\Phi_j,\mathbf{v},\mathbf{v})}{\|\mathbf{v}\|^2} > 0$$

Following the construction described in the subsection 4.3 and according to [19, 29], under the stability of the wavelet basis Ψ_i, $i = 0,\ldots,j$, a preconditioner $\tilde{\mathbf{P}}_{\tau,\nu;\Psi}(\mathbf{y}_j)$ can be defined by

$$\tilde{\mathbf{P}}_{\tau,\nu;\Psi}(\mathbf{y}_j) = \mathbf{L}_j^{-1}\tilde{\mathbf{D}}_{\tau,\nu;j}(\mathbf{y}_j)\mathbf{L}_j,$$

where $\tilde{\mathbf{D}}_{\tau,\nu;j}(\mathbf{y}_j)$ is a diagonal block matrix with diagonal blocks $\tilde{\mathbf{D}}^n_{\tau,\nu;j}(\mathbf{y}_j), n = 1,\ldots,d$, which entries are given by

$$\left(\tilde{\mathbf{D}}^n_{\tau,\nu;j}(\mathbf{y}_j)\right)_{(i,i'),(k,k')} = d_i(\mathbf{y}_j^n)\delta_{i,i'}\delta_{k,k'} \quad i,i' = 0,\ldots,j \quad k,k' \in \Lambda_i$$

for certain positive constants $d_i(\mathbf{y}_j^n), n = 1,\ldots,d$.

We are interested in estimate the condition number of $\tilde{\mathbf{P}}_{\tau,\nu;\Psi}(\mathbf{y}_j)^{-1}\tilde{\mathbf{F}}_{\tau,\nu;\Psi}(\mathbf{y}_j)$, that is

$$\kappa\left(\tilde{\mathbf{P}}_{\tau,\nu;\Psi}(\mathbf{y}_j)^{-1}\tilde{\mathbf{F}}_{\tau,\nu;\Psi}(\mathbf{y}_j)\right) \leq C\frac{\Gamma_j}{\gamma_j}, \tag{54}$$

where the constants γ_j and Γ_j comes from the inequality (50).

Let us remark that our matrix looks like the expansion

$$\tilde{\mathbf{D}}^n_{\tau,\nu;\Psi}(\mathbf{y}_j)\mathbf{v}^n = \sum_{i=0}^{j} d_i(\mathbf{y}^n_j) \sum_{k\in\Lambda_i} \langle \mathbf{v}^n, \tilde{\psi}^n_{i,k}\rangle \psi^n_{i,k}$$

and

$$\left\langle \tilde{\mathbf{D}}_{\tau,\nu;\Psi}(\mathbf{y}_j)\mathbf{v}, \mathbf{v}\right\rangle := \sum_{n=1}^{d} \left\langle \tilde{\mathbf{D}}^n_{\tau,\nu;\Psi}(\mathbf{y}_j)\mathbf{v}^n, \mathbf{v}^n\right\rangle .$$

The operator $\mathbf{A}_{\tau,\nu;\Psi}$ is a diagonal block matrix, with blocks $\mathbf{A}^n_{\tau,\nu;\Psi}, n = 1,\ldots,d$, where each block is a linear combination of stiffness and mass matrices. Therefore, we can conclude by c.f. [20] that each block is spectrally equivalent to $\tau+\nu 2^{2jd}$. Then we have the spectral equivalent relation for each block $\mathbf{A}^n_{\tau,\nu;\Psi}$

$$\frac{\langle \mathbf{A}^n_{\tau,\nu;\Psi}\psi^n_{i,k}, \psi^n_{i,k}\rangle}{\|\psi^n_{i,k}\|^2} \sim \tau + \nu 2^{2jd} \tag{55}$$

The matrices $\tilde{\mathbf{N}}_{p;\Psi}(\mathbf{y}_j^T\Phi_j), p = 1,2$, are diagonal blocks matrices with diagonal blocks $\tilde{\mathbf{N}}^n_{p;\Psi}(\mathbf{y}_j^T\Phi_j), n = 1,\ldots,d$.

Let us take a close look for the diagonal entries of $\tilde{\mathbf{N}}^n_{p;\Psi}(\mathbf{y}_j^T\Phi_j)$, $p = 1,2$,

$$\sum_{m=1}^{d} \pi^n_j \left(\mathbf{y}^{mT}_j \Phi^m_j \frac{\partial\psi^n_{i,k}}{\partial x_m}\psi^n_{i,k}\right)$$

and

$$\sum_{m=1}^{d} \pi^n_j \left(\psi^m_{i,k} \frac{\partial(\mathbf{y}^{nT}_j\Phi^n_j)}{\partial x_m}\psi^n_{i,k}\right),$$

respectively. We will make estimates using knot oriented quadrature rules developed in the subsection 3.2.

Let us remark that

$$\|\psi^n_{i,k}\|^2 \approx 2^{-jd} \sum_{l\in\Lambda_j} |\psi^n_{i,k}(2^{-j}l)|^2,$$

$$\frac{1}{2}\int_\Omega \frac{\partial|\psi^n_{i,k}|^2}{\partial x_m}dx = \int_\Omega \frac{\partial\psi^n_{i,k}}{\partial x_m}\psi^n_{i,k}dx \approx 2^{-jd}\sum_{l\in\Lambda_j} \frac{\partial\psi^n_{i,k}}{\partial x_m}(2^{-j}l)\psi^n_{i,k}(2^{-j}l),$$

Then we have

$$\min_{l\in\Lambda_j}\sum_{m=1}^{d} \frac{\partial(\mathbf{y}^{nT}_j\Phi^n_j)}{\partial x_m}(2^{-j}l) \le \frac{\sum_{m=1}^{d}\pi^n_j\left(\psi^m_{i,k}\frac{\partial(\mathbf{y}^{nT}_j\Phi^n_j)}{\partial x_m}\psi^n_{i,k}\right)}{\|\psi^n_{i,k}\|^2} \le \max_{l\in\Lambda_j}\sum_{m=1}^{d} \frac{\partial(\mathbf{y}^{nT}_j\Phi^n_j)}{\partial x_m}(2^{-j}l) \tag{56}$$

$$\left|\sum_{m=1}^{d} \pi^n_j \left(\mathbf{y}^{mT}_j \Phi^m_j \frac{\partial\psi^n_{i,k}}{\partial x_m}\psi^n_{i,k}\right)\right| \le 2^{jd/2}\max_{l\in\Lambda_j}|\mathbf{y}^n_{j,l}| \left|\sum_{m=1}^{d}\int_\Omega \frac{\partial|\psi^n_{j,k}|^2}{\partial x_m}dx\right|. \tag{57}$$

Expanding $\frac{\partial(\mathbf{y}_j^{n\,T}\Phi_j^n)}{\partial x_m}$ in a bi-orthogonal expansion and using the interpolating property and the identity $\langle \frac{\partial \phi_{j,p}^n}{\partial x_m}, \tilde{\phi}_{j,l}^n \rangle = 2^j g_{p,l}$, where

$$g_{p,l} := \int_\Omega \frac{\partial \phi^n}{\partial x_m}(x-p)\tilde{\phi}^n(x-l)dx$$

is a constant independent of j, m and n. Then we have

$$\frac{\partial(\mathbf{y}_j^{n\,T}\Phi_j^n)}{\partial x_m}(2^{-j}l) = 2^{j(d/2+1)} \sum_{p\in\Lambda_j} g_{p,l} y_{j,p}^n.$$

Moreover, the inequality (56) becomes then

$$\frac{\sum_{m=1}^d \pi_j^n \left(\psi_{i,k}^m \frac{\partial(\mathbf{y}_j^{n\,T}\Phi_j^n)}{\partial x_m} \psi_{i,k}^n\right)}{\|\psi_{i,k}^n\|^2} \geq d2^{j(d/2+1)} \min_{l\in\Lambda_j} \sum_{p\in\Lambda_j} g_{p,l} y_{j,p}^n \tag{58}$$

$$\frac{\sum_{m=1}^d \pi_j^n \left(\psi_{i,k}^m \frac{\partial(\mathbf{y}_j^{n\,T}\Phi_j^n)}{\partial x_m} \psi_{i,k}^n\right)}{\|\psi_{i,k}^n\|^2} \leq d2^{j(d/2+1)} \max_{l\in\Lambda_j} \sum_{p\in\Lambda_j} g_{p,l} y_{j,p}^n \tag{59}$$

On the other hand, using Stokes's theorem, under zeroth Dirichlet boundary conditions, we have obtain

$$\sum_{m=1}^d \int_\Omega \frac{\partial |\psi_{i,k}^n|^2}{\partial x_m} dx = \int_\Gamma |\psi_{i,k}^n|^2 d\Gamma = 0.$$

Therefore, using relation (57), we obtain

$$\frac{\sum_{m=1}^d \pi_j^n \left(\mathbf{y}_j^{m\,T}\Phi_j^m \frac{\partial \psi_{i,k}^n}{\partial x_m} \psi_{i,k}^n\right)}{\|\psi_{i,k}^n\|^2} \approx 0.$$

Taking

$$d_i(\mathbf{y}_j^n) = \nu 2^{2id} + d2^{j(d/2+1)} \min_{l\in\Lambda_j} \sum_{p\in\Lambda_j} g_{p,l} y_{j,p}^n,$$

we have that

$$\gamma_{i,j}^n \left\langle \tilde{\mathbf{D}}_{\tau,\nu;\Psi}^n(\mathbf{y}_j)\psi_{i,k}^n, \psi_{i,k}^n \right\rangle \leq \left\langle \tilde{\mathbf{F}}_{\tau,\nu;\Psi}^n(\mathbf{y}_j)\psi_{i,k}^n, \psi_{i,k}^n \right\rangle \leq \Gamma_{i,j}^n \left\langle \tilde{\mathbf{D}}_{\tau,\nu;\Psi}^n(\mathbf{y}_j)\psi_{i,k}^n, \psi_{i,k}^n \right\rangle,$$

with

$$\gamma_{i,j}^n = \frac{\tau + \nu 2^{2id} + d2^{j(d/2+1)} \min_{l\in\Lambda_j} \sum_{p\in\Lambda_j} g_{p,l} y_{j,p}^n}{\nu 2^{2id} + d2^{j(d/2+1)} \min_{l\in\Lambda_j} \sum_{p\in\Lambda_j} g_{p,l} y_{j,p}^n}$$

and

$$\Gamma_{i,j}^n = \frac{\tau + \nu 2^{2id} + d2^{j(d/2+1)} \max_{l\in\Lambda_j} \sum_{p\in\Lambda_j} g_{p,l} y_{j,p}^n}{\nu 2^{2id} + d2^{j(d/2+1)} \min_{l\in\Lambda_j} \sum_{p\in\Lambda_j} g_{p,l} y_{j,p}^n}.$$

Summing up for all $n = 1, \ldots, d$, we have

$$\begin{aligned}\min_{n=1,\ldots,d} \gamma_{i,j}^n \left\langle \tilde{\mathbf{D}}_{\tau,\nu;\Psi}(\mathbf{y}_j)\psi_{i,k}, \psi_{i,k}\right\rangle &\leq \left\langle \tilde{\mathbf{F}}_{\tau,\nu;\Psi}(\mathbf{y}_j)\psi_{i,k}, \psi_{i,k}\right\rangle \\ &\leq \max_{n=1,\ldots,d} \Gamma_{i,j}^n \left\langle \tilde{\mathbf{D}}_{\tau,\nu;\Psi}(\mathbf{y}_j)\psi_{i,k}, \psi_{i,k}\right\rangle .\end{aligned}$$

Because each element of X_j can be expressed as a linear combination of $\psi_{i,k}$, we prove that

$$\gamma_j \left\langle \tilde{\mathbf{D}}_{\tau,\nu;\Psi}(\mathbf{y}_j)\mathbf{v}, \mathbf{v}\right\rangle \leq \left\langle \tilde{\mathbf{F}}_{\tau,\nu;\Psi}(\mathbf{y}_j)\mathbf{v}, \mathbf{v}\right\rangle \leq \Gamma_j \left\langle \tilde{\mathbf{D}}_{\tau,\nu;\Psi}(\mathbf{y}_j)\mathbf{v}, \mathbf{v}\right\rangle$$

holds for all $\mathbf{v} \in X_j$, where

$$\begin{aligned}\gamma_j &= \min_{i=0,\ldots,j} \min_{n=1,\ldots,d} \gamma_{i,j}^n \\ \Gamma_j &= \max_{i=0,\ldots,j} \max_{n=1,\ldots,d} \Gamma_{i,j}^n\end{aligned}$$

Thus, we prove that $\tilde{\mathbf{P}}_{\tau,\nu;\Psi}(\mathbf{y}_j) = \mathbf{L}_j^{-1}\tilde{\mathbf{D}}_{\tau,\nu;\Psi}(\mathbf{y}_j)\mathbf{L}_j$ is a preconditioner for $\tilde{\mathbf{F}}_{\tau,\nu;\Psi}(\mathbf{y}_j)$ which satisfy the conditions imposed in subsection 4.3 and condition (54). Furthermore, the condition number of $\tilde{\mathbf{P}}_{\tau,\nu;\Psi}(\mathbf{y}_j)^{-1}\tilde{\mathbf{F}}_{\tau,\nu;\Psi}(\mathbf{y}_j)$ is bounded by $\frac{\Gamma_j}{\gamma_j}$.

Furthermore, our preconditioner is optimal if $\min_{l\in\Lambda_j} \sum_{p\in\Lambda_j} g_{p,l} y_{j,p}^n$ is closed to $\max_{l\in\Lambda_j} \sum_{p\in\Lambda_j} g_{p,l} y_{j,p}^n$ as j increases.

6 Numerical Examples

To confirm the applicability of our approach we present a test example for the two dimensional case. Numerical examples results were obtained by choosing the (periodized) Deslauriers-Dubuc interpolating scaling functions of order 2,4 and 6, respectively.

For this purpose, we assume that our exact solution $(\mathbf{u}, p) = (u_1, u_2, p)$ is given by

$$\begin{aligned}u_1(x_1, x_2) &= \cos(2\pi x_1)\sin(2\pi x_2) \\ u_2(x_1, x_2) &= -\sin(2\pi x_1)\cos(2\pi x_2) \\ p(x_1, x_2) &= \sin(2\pi x_1)\sin(2\pi x_2)\end{aligned}$$

which is a sufficiently smooth periodic function on $\Omega = [0, 1]^2$. For the kinematic viscosity and for the time step, we choose $\nu = 20$ and $\tau = 1$, respectively. Let us remark that our exact solution $\mathbf{u}$ satisfy the divergence free condition ($\operatorname{div} \mathbf{u} = 0$).

The exact solution is displayed on the figure 1, the approximation of the u_1, u_2, and p and its error approximation are displayed on the figure $2, 3$ and 4.

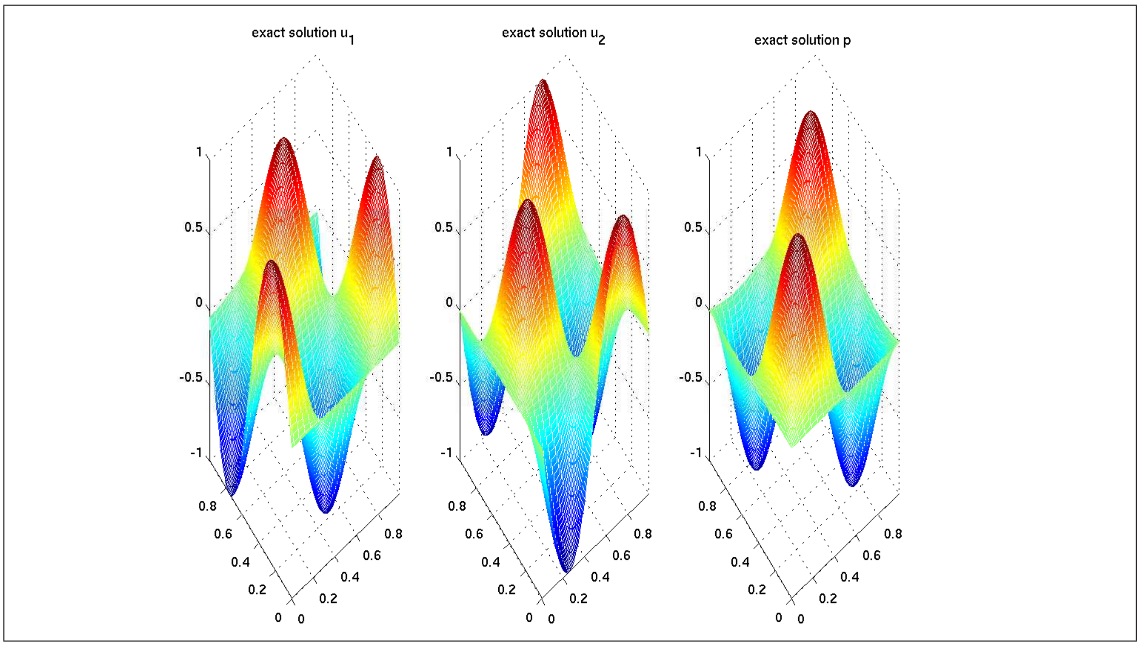

Figure 1: Exact solution for the Navier-Stokes equations.

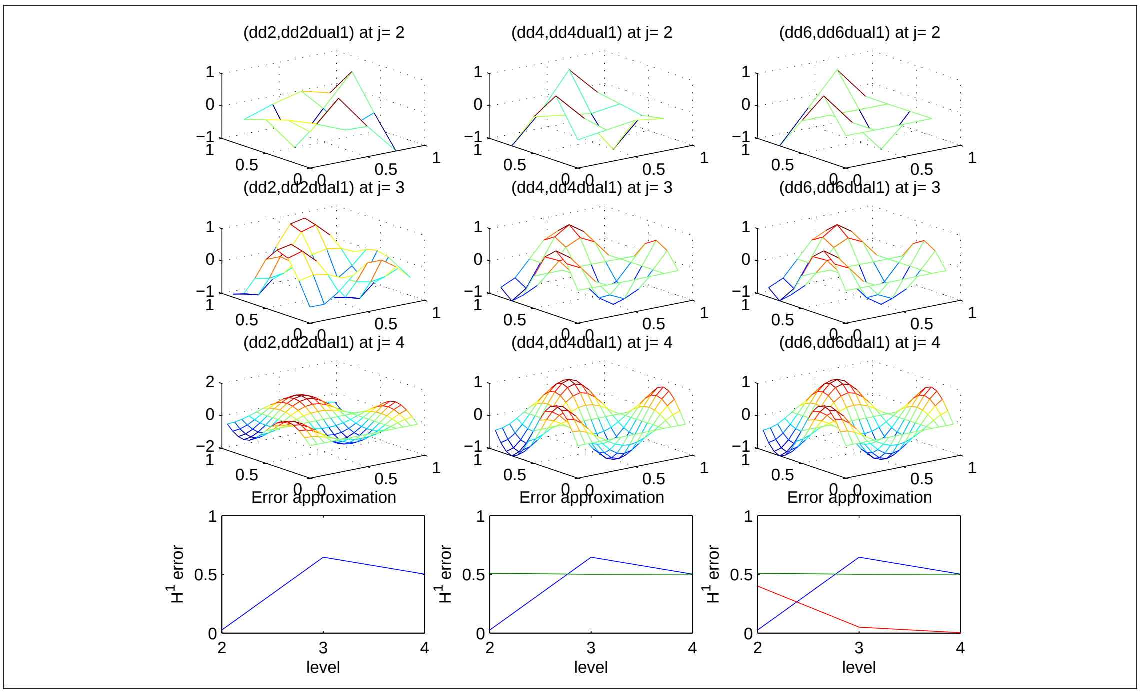

Figure 2: Approximation of u_1 for the levels $j = 2, 3, 4$ and error approximation.

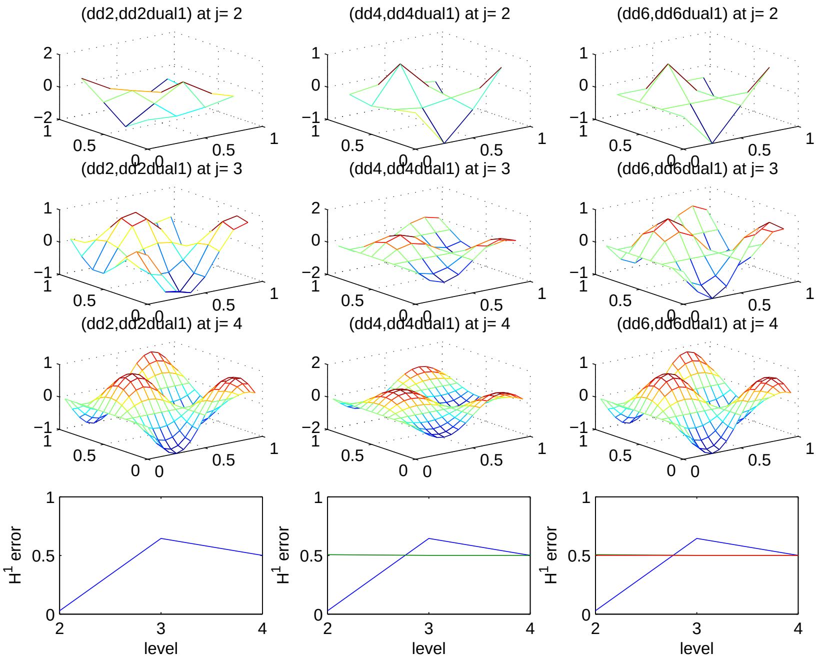

Figure 3: Approximation of u_2 for the levels $j = 2, 3, 4$ and error approximation.

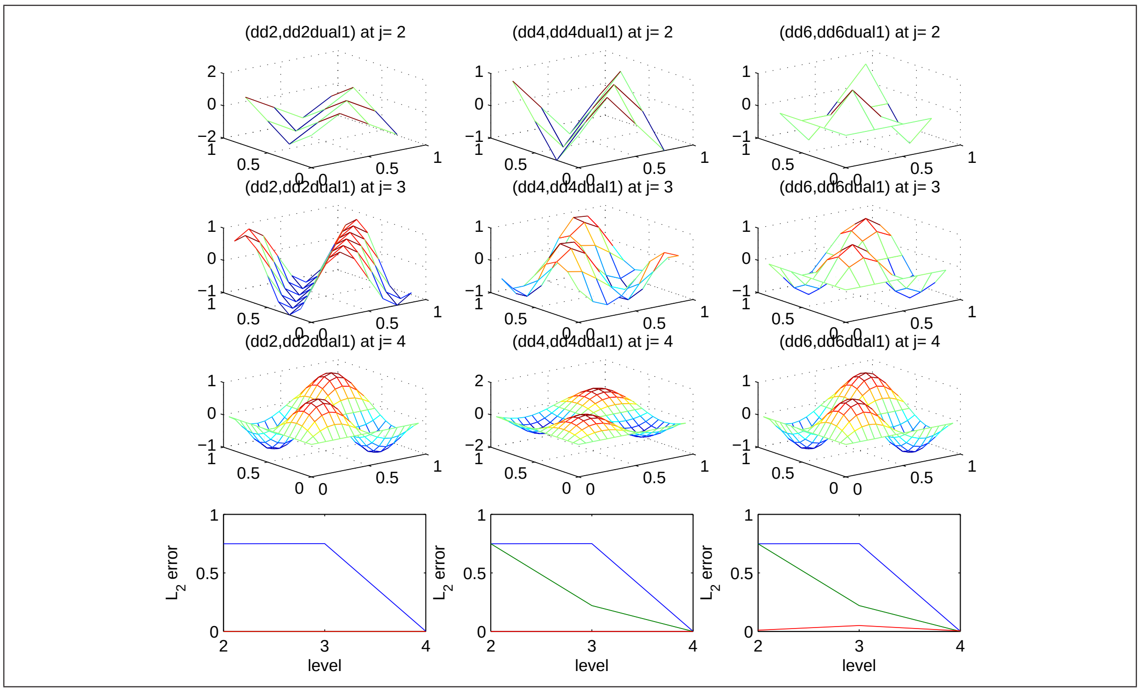

Figure 4: Approximation of p for the levels $j = 2, 3, 4$ and error approximation.

Conclusion: As we can see in the figures $2, 3$ and 4, the error approximation depends on the level and on the order of the wavelets. Hence, the increasing of the order of the Deslauriers-Dubuc wavelets provides us to get better accurate results.

References

[1] Adams,R.A., *Sobolev spaces*, Pure and Applied Mathematics-Series, Academic Press, Boston-New York-London, 1978.

[2] Barinka, A., Barsch, T., Charton, P., Cohen, A., Dahlke, S., Dahmen, W., Urban, K.: *Adaptive wavelet schemes for elliptic problems-Implementation and numerical experiments*. SIAM J. Scientific Comput. 23(3), 910-939 (2001).

[3] Benzi, M., Golub, G.H. and Liesen, J., *Numerical solution of saddle point problems*, Acta Numerica 2005, 10:1-137

[4] Böhmer, K. and Dahlke, S.: *On the Stability of Multiscale Wavelet Methods with Applications to Navier-Stokes Equations.*, International Journal of Pure and Applied Mathematics 6(4) (2003), 465-489.

[5] Braess D., *Finite Elements: Theory, fast solvers, and applications in solids mechanics*, Second edition, Cambbridge University press, 2001.

[6] Bramble, J.H., Pasciak, J.E.: *A preconditioning technique for indefinite systems resulting from mixed approximations for elliptic problems.*, Math. Comp. 50 (1988), 1-17.

[7] Bramble, J.H., Pasciak, J.E., Vassilev, A.T.: *Uzawa type algorithms for nonsymmetric saddle point problems.*, Math. Comp. 69 (2000), 667-689.

[8] Brezzi, F. and Fortin, M.: *Mixed and Hybrid Finite Element Methods.*, Springer (1991).

[9] Caloz,G. and Rappaz, J. *Numerical Analysis for Nonlinear and Bifurcation Problems.*, In P.G.Ciarlet and J.L.Lions, editors, Handbook of Numerical Analysis, Volume V, pages 487-637. Elsevier 1997.

[10] Cerejeiras, P., Dahlke, S., Kähler, U., Lindemann, M., Soares, M.J., Teschke, G. and Zhariy, M. *A Wavelet Galerkin Scheme for Nonlinear Elliptic Partial Differential Equations*, IKM Proceedings, Weimar, 2003

[11] Cohen, A. Dahmen, W., DeVore, R.: *Adaptive Wavelet Methods II-Beyond the Elliptic Case*, Found. Comput. Math. 2 (2002), 203-245.

[12] Cohen, A. Dahmen, W., DeVore, R.: *Adaptive Wavelet Methods for Nonlinear Variational Problems*, SIAM J. Numer. Anal. 41 (5) (2003), 1785-1823.

[13] Dahlke, S., Gröchenig, K., and Maass, P., A new approach to interpolating scaling functions, *Appl. Anal.* **72(3–4)** (1999), 485–500.

[14] Dahlke, S., Dahmen, W., Urban, K., *Adaptive Wavelet Methods for Saddle Point Problems - Optimal Convergence Rates* , SIAM Journal on Numerical Analysis 40, No. 4 (2002)1230-1262.

[15] Dahlke, S., Maass, P. and Teschke, G. Interpolating Scaling Functions with Duals, *Journal of Computational Analysis and Applications*, 5(3), (2003), 361-373.

[16] Dahmen, W., *Some remarks on multiscale transformations, stability and biorthogonality*, in: Curves and Surfaces II, P. J. Laurent, A. Le Méhauté, and L. L. Schumaker (eds.), AKPeters Wellesley, Boston, 1994, 157-188.

[17] Dahmen, W., *Wavelets and multiscale methods for operator equations*, Acta Numerica 1997, 6: 55-228

[18] Dahmen, W., Prössdorf, S., Schneider, R., *Multiscale methods for pseudo-differential equations on smooth manifolds.* in: Proceedings of the International Conference on Wavelets: Theory, Algorithms, and Applications, C.K. Chui, L. Montefusco, L. Puccio (eds.), Academic Press, 385-424, 1994.

[19] Dahmen, W., Kunoth, A.,*Multilevel Preconditioning*, Numerische Mathematik **63**, (1992), 315-344.

[20] Dahmen, W., Kunoth, A., Urban, K., *A Wavelet-Galerkin Method for the Stokes Equations* , Computing 56, No. 3, 1996, 259-302

[21] Dahmen, W., Urban, K. and Vorloeper, J., *Adaptive Wavelet Methods – Basic Concepts and Applications to the Stokes Problem*, Wavelet Analysis – Twenty Years Developments, Ding–Xuan Zhou ed., World Scientific, New Jersey, (2002), 39–80.

[22] Dahmen, W., Kunoth, A., Urban, K., *Biorthogonal spline wavelets on the interval – Stability and moment conditions*, Preprint, RWTH Aachen, 1996.

[23] Derado, J., Multivariate refinable interpolating functions, Preprint, University of Connecticut, (1997).

[24] Deslauriers, G., Dubois, J. and Dubuc, S. Multidimensional iterative interpolation, *Can. J. Math.* **43** (1991), 297–312.

[25] Dubuc, S. Interpolation through an iterative scheme, *J. Math. Anal. Appl.* **114** (1986), 185–204.

[26] Deslauriers, G. and Dubuc, S. Interpolation dyadique, in: *Fractals, Dimensions non Entières et Applications*, (G. Cherbit, Ed.), Masson, Paris, 1987, pp. 44–45.

[27] Girault, V. and Raviart, P.-A.: *Finite Element Methods for Navier-Stokes Equations.*, Springer (1986).

[28] H. Ji, S.D. Riemenschneider, and Z. Shen, Multivariate compactly supported fundamental refinable functions, dual and biorthogonal wavelets, to appear in: *Studies in Applied Math.*

[29] Kunoth, A. *Multilevel Preconditioning - Appending Boundary Conditions by Lagrange Multipliers*, Adv. Comput. Math. **4**, (1995), 145-170

[30] A. Quarteroni, A. Valli, Numerical approximation of partial differential equations, Springer, Heidelberg, 1994.

[31] S.D. Riemenschneider and Z. Shen, Multidimensional interpolatory subdivision schemes, *SIAM J. Numer. Anal.* **34** (1997), 2357–2381.

[32] Teman, R: *Navier-Stokes equations, theory and numerical analysis.*, AMS Chelsea Publishing, New York, 2001-*reprint*.